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Equitable colorings of Cartesian products of graphs[☆]

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ABSTRACT

The present paper studies the following variation of vertex coloring on graphs. A graph G is equitably k-colorable if there is a mapping $f: V(G) \to \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ for $xy \in E(G)$ and $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$ for $1 \leq i, j \leq k$. The equitable chromatic number of a graph G, denoted by $\chi_=(G)$, is the minimum k such that G is equitably k-colorable. The equitable chromatic threshold of a graph G, denoted by $\chi_=^*(G)$, is the minimum t such that G is equitably k-colorable for all $k \geq t$. Our focus is on the equitable colorability of Cartesian products of graphs. In particular, we give exact values or upper bounds of $\chi_=(G\Box H)$ and $\chi_=^*(G\Box H)$ when G and H are cycles, paths, stars, or complete bipartite graphs.

1. Introduction

Graph coloring plays a central position in discrete mathematics. During the past hundred years, many deep and interesting results have been obtained, and various applications have arisen. In the current paper, we focus on a restricted version of graph coloring called equitable coloring.

For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$. A (*proper*) *k-coloring* of a graph G is a mapping $f: V(G) \to [k]$ such that $f(x) \neq f(y)$ for $xy \in E(G)$. We call the set $f^{-1}(i) = \{x \in V(G): f(x) = i\}$ a *color class* for $i \in [k]$. Notice that each color class is an independent set, i.e., a pairwise non-adjacent vertex set. A graph is *k-colorable* if it has a k-coloring. The *chromatic number* of G is $\chi(G) = \min\{k: G \text{ is } k\text{-colorable}\}$.

This paper focuses on the following variation of coloring. An *equitable k-coloring* is a k-coloring for which any two color classes differ in size by at most 1. If a graph of n vertices is equitably k-colorable then each color class is of size $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$; more precisely, the color classes have sizes $\lfloor \frac{n+i-1}{k} \rfloor \left(= \lceil \frac{n-k+i}{k} \rceil\right)$ for $i \in [k]$. The *equitable chromatic number* of G is $\chi_=(G) = \min\{k: G \text{ is equitably } k\text{-colorable}\}$ and the *equitable chromatic threshold* of G is $\chi_=^*(G) = \min\{t: G \text{ is equitably } k\text{-colorable for all } k \geq t\}$. The concept of equitable colorability was first introduced by Meyer [26]. His motivation came from the application given by Tucker [32] where vertices represented garbage collection routes and two such vertices were joined when the corresponding routes should not be run on the same day. For more applications such as scheduling and constructing timetables, please see [1,12,13,16,28,31,32]. For a good survey, please see the paper by Lih [23].

In 1964 Erdős [7] conjectured that any graph G with maximum degree $\Delta(G) \leq k$ has an equitable $(k+1)$-coloring. This conjecture was proved in 1970 by Hajnal and Szemerédi [9] with a long and complicated proof. Mydlarz and Szemerédi [27] found a polynomial-time algorithm for such a coloring. Recently, Kierstead and Kostochka [14] gave a short proof of the

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theorem, and presented another polynomial-time algorithm for such a coloring. They [15] proved an even stronger result that every graph satisfying $d(x) + d(y) \le 2k + 1$ for every edge xy has an equitable $(k + 1)$-coloring. Brooks type results are conjectured: the Equitable Coloring Conjecture [26] $\chi_=(G) \le \Delta(G)$, and the Equitable Δ-Coloring Conjecture [5] $\chi^*_=(G) \le \Delta(G)$ for $G \notin \{K_n, C_{2n+1}, K_{2n+1,2n+1}\}$. Exact values of equitable chromatic numbers and equitable thresholds of trees [3,4] and complete multipartite graphs [2,22] were determined. Chen et al. [6] and Furmańczyk [8] investigated equitable colorability of square and cross products of graphs. Equitable coloring has been extensively studied in the literature; see [4,5,17–21, 23–25,28,29,34–36].

Among the known results on equitable coloring, we are most interested in those on graph products. Notice that studying the relation of graph parameters between the product and its factors is helpful for analyzing the structure of complicated graphs; see [10,11,30,33,37]. The *Cartesian* (or *square*) *product* of graphs G and H is the graph $G\Box H$ with vertex set $\{(x, y): x \in V(G), y \in V(H)\}$ and edge set $\{(x, y)(x', y'): x = x'$ with $yy' \in E(H)$ or $xx' \in E(G)$ with $y = y'\}$.

This paper is organized as follows. Section 2 is a review for equitable colorings on Cartesian products of graphs related to our results in this paper. Section 3 establishes exact values of equitable chromatic numbers and thresholds of Cartesian products of an odd cycle or an odd path with a bipartite graph, an even cycle or an even path with a complete bipartite graph, and two stars; and upper bounds on the equitable chromatic number and threshold of the Cartesian product of two complete bipartite graphs. In the last section, we summarize our results and give some open problems.

2. Preliminaries

For an integer positive n, the *n-path* is the graph P_n with vertex set $\{x_1, x_2, \dots, x_n\}$ and edge set $\{x_1x_2, x_2x_3, \dots, x_{n-1}x_n\}$. For an integer $n \ge 3$, the *n-cycle* is the graph C_n with vertex set $\{x_1, x_2, \dots, x_n\}$ and edge set $\{x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_nx_1\}$. For positive integers m and n, the *complete bipartite graph* $K_{m,n}$ is the graph with vertex set $\{y_i, z_j : 1 \le i \le m, 1 \le j \le n\}$ and edge set $\{y_iz_j : 1 \le i \le m, 1 \le j \le n\}$. A *bipartite graph* is a subgraph of a complete bipartite graph.

It is evident from the definition that $\chi(G) \le \chi_=(G) \le \chi^*_=(G)$ for any graph G. In general, the inequalities can be strict. For example,

$$\chi(K_{1,4}) = 2 < \chi_=(K_{1,4}) = \chi^*_=(K_{1,4}) = 3,$$
$$\chi(K_{3,3}) = \chi_=(K_{3,3}) = 2 < \chi^*_=(K_{3,3}) = 4,$$
$$\chi(K_{5,8}) = 2 < \chi_=(K_{5,8}) = 3 < \chi^*_=(K_{5,8}) = 5.$$

The following result by Chen et al. [6] is of most interest in our study on equitable colorability for Cartesian products of graphs.

Theorem 1 ([6]). *If G and H are equitably k-colorable, then so is $G\Box H$.*

Consequently, we have the following inequality for the equitable chromatic threshold:

Corollary 2. $\chi^*_=(G\Box H) \le \max\{\chi^*_=(G), \chi^*_=(H)\}$.

Corollary 3. *If G and H are graphs with $\chi(G) = \chi^*_=(G)$ and $\chi(H) = \chi^*_=(H)$, then $\chi(G\Box H) = \chi_=(G\Box H) = \chi^*_=(G\Box H) = \max\{\chi(G), \chi(H)\}$.*

Proof. The result follows from Corollary 2 and Sabidussi's result [30] that $\chi(G\Box H) = \max\{\chi(G), \chi(H)\}$. □

Examples of graphs G with $\chi(G) = \chi^*_=(G)$ include complete graphs, paths and cycles; see [6,8] for Corollary 3 on these three classes of graphs. For instance, $\chi(K_m\Box K_n) = \chi_=(K_m\Box K_n) = \chi^*_=(K_m\Box K_n) = \max\{m, n\}$, $\chi(C_m\Box C_n) = \chi_=(C_m\Box C_n) = \chi^*_=(C_m\Box C_n) = 2$ (resp. 3) if m and n are even (resp. m or n is odd), and $\chi_=(K_{1,m}\Box P_n) = 3$ for $m \ge 3$ and odd $n \ge 3$.

Unlike the equitable chromatic threshold, $\chi_=(G\Box H) \le \max\{\chi_=(G), \chi_=(H)\}$ is false in general. For instance, Chen et al. [6] showed that $\chi_=(K_{1,1,2}\Box K_{3,3}) = 4 > \max\{\chi_=(K_{1,1,2}), \chi_=(K_{3,3})\} = 3$. They [6] also mentioned that $\chi_=(G) = \chi_=(H) = k$ may not lead to $\chi_=(G\Box H) = k$ with the example $\chi_=(K_{1,2n}) = n + 1$ while $\chi_=(K_{1,2n}\Box K_{1,2n}) \le 4$. In Section 3 we shall give more general results of this kind.

3. The Cartesian product of graphs

We now study equitable chromatic numbers and equitable chromatic thresholds of Cartesian products of graphs for three cases as follows.

3.1. The product of $C_{2\ell+1}$ or $P_{2\ell+1}$ with a bipartite graph

We first study the Cartesian product of an odd cycle or an odd path with a bipartite graph.

Theorem 4. *If ℓ is a positive integer and H is a bipartite graph, then $\chi_=(C_{2\ell+1}\Box H) = \chi^*_=(C_{2\ell+1}\Box H) = \chi_=(P_{2\ell+1}\Box H) = \chi^*_=(P_{2\ell+1}\Box H) = 3$ except that $\chi_=(P_{2\ell+1}\Box H) = \chi^*_=(P_{2\ell+1}\Box H) = 2$ for the case when $\chi_=(H) \le 2$.*

Proof. Recall that the vertex set of $C_{2\ell+1}$ or $P_{2\ell+1}$ is $\{x_1, x_2, \dots, x_{2\ell+1}\}$. Suppose the bipartition of the graph H consists of $Y = \{y_1, y_2, \dots, y_m\}$ and $Z = \{z_1, z_2, \dots, z_n\}$. We order the vertices of the product graph $C_{2\ell+1}\Box H$ or $P_{2\ell+1}\Box H$ as in Fig. 1. Notice that any set consisting of consecutive vertices in the ordering of size no more than $\ell(m + n)$ is an independent set.

$(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_m), (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n),$
$(x_3, y_1), (x_3, y_2), \ldots, (x_3, y_m), (x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), \ldots, (x_{2\ell+1}, y_1), (x_{2\ell+1}, y_2), \ldots, (x_{2\ell+1}, y_m),$
$(x_1, z_1), (x_1, z_2), \ldots, (x_1, z_n), (x_2, y_1), (x_2, y_2), \ldots, (x_2, y_m),$
$(x_3, z_1), (x_3, z_2), \ldots, (x_3, z_n), (x_4, y_1), (x_4, y_2), \ldots, (x_4, y_m), \ldots, (x_{2\ell+1}, z_1), (x_{2\ell+1}, z_2), \ldots, (x_{2\ell+1}, z_n).$

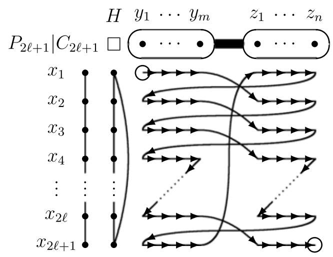

Fig. 1. Vertex ordering for the product graph.

For $k \geq 3$, let $\sigma_t = \lfloor \frac{(2\ell+1)(m+n)+t-1}{k} \rfloor$ for $t \in [k]$. Since $\sigma_k = \lfloor \frac{(2\ell+1)(m+n)+k-1}{k} \rfloor \leq \ell(m+n)$, we can partition the vertex set of the product graph into k independent sets of sizes $\sigma_1, \sigma_2, \ldots, \sigma_k$ consecutively in the ordering. Hence the product graph is equitably k-colorable.

On the other hand, we have that $\chi_=(C_{2\ell+1}\square H) \geq \chi(C_{2\ell+1}\square H) \geq \chi(C_{2\ell+1}) = 3$. Also, the bipartite graph $P_{2\ell+1}\square H$ is equitably 2-colorable if and only if its vertex set can be divided into two partition sets which differ by at most 1 in size, or equivalently, $|(\ell m + (\ell+1)n) - ((\ell+1)m + \ell n)| = |n - m| \leq 1$. The theorem then follows. $\square$

3.2. *The product of $C_{2\ell}$ or $P_{2\ell}$ with a complete bipartite graph*

Next we study the Cartesian product of an even cycle or an even path with a bipartite graph. In this case, we are only able to establish results for $C_{2\ell}\square K_{m,n}$ and $P_{2\ell}\square K_{m,n}$.

Lemma 5. *If ℓ, m and n are positive integers with $m \leq n$, then $C_{2\ell}\square K_{m,n}$ and $P_{2\ell}\square K_{m,n}$ are equitably k-colorable for $k \geq 2$, except that $C_4\square K_{m,n}$ and $P_2\square K_{m,n}$ may or may not be equitably 3-colorable.*

Proof. We observe that $C_{2\ell}\square K_{m,n}$ and $P_{2\ell}\square K_{m,n}$ are bipartite graphs whose bipartitions consist of $\mathcal{V}_1 = \{(x_{2i-1}, y_j), (x_{2i}, z_{j'}) : i \in [\ell], j \in [m], j' \in [n]\}$ and $\mathcal{V}_2 = \{(x_{2i}, y_j), (x_{2i-1}, z_{j'}) : i \in [\ell], j \in [m], j' \in [n]\}$ of the same size $\ell(m+n)$. Hence, they are equitably 2-colorable.

We order the vertices of the product graph $C_{2\ell}\square K_{m,n}$ as in Fig. 2 and $P_{2\ell}\square K_{m,n}$ as in Fig. 3. Notice that among those sets consisting of consecutive vertices in the orderings, the largest independent set containing $(x_{2\ell}, z_n)$ and (x_2, y_1) in $C_{2\ell}\square K_{m,n}$ is of size $(\ell-1)(m+n)$, and the one containing $(x_{2\ell}, z_n)$ and (x_1, z_1) in $P_{2\ell}\square K_{m,n}$ is of size $\ell(m+n) - m$.

$(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_m), (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n),$
$(x_3, y_1), (x_3, y_2), \ldots, (x_3, y_m), (x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), \ldots, (x_{2\ell}, z_1), (x_{2\ell}, z_2), \ldots, (x_{2\ell}, z_n),$
$(x_2, y_1), (x_2, y_2), \ldots, (x_2, y_m), (x_3, z_1), (x_3, z_2), \ldots, (x_3, z_n),$
$(x_4, y_1), (x_4, y_2), \ldots, (x_4, y_m), (x_5, z_1), (x_5, z_2), \ldots, (x_5, z_n),$
$\ldots, (x_{2\ell}, y_1), (x_{2\ell}, y_2), \ldots, (x_{2\ell}, y_m), (x_1, z_1), (x_1, z_2), \ldots, (x_1, z_n).$

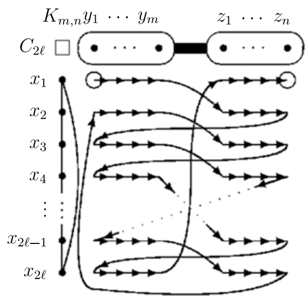

Fig. 2. Vertex ordering for $C_{2\ell}\square K_{m,n}$.

$$(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_m), (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n),$$
$$(x_3, y_1), (x_3, y_2), \ldots, (x_3, y_m), (x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), \ldots, (x_{2\ell}, z_1), (x_{2\ell}, z_2), \ldots, (x_{2\ell}, z_n),$$
$$(x_1, z_1), (x_1, z_2), \ldots, (x_1, z_n), (x_2, y_1), (x_2, y_2), \ldots, (x_2, y_m),$$
$$(x_3, z_1), (x_3, z_2), \ldots, (x_3, z_n), (x_4, y_1), (x_4, y_2), \ldots, (x_4, y_m), \ldots, (x_{2\ell}, y_1), (x_{2\ell}, y_2), \ldots, (x_{2\ell}, y_m).$$

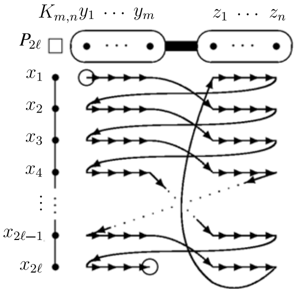

Fig. 3. Vertex ordering for $P_{2\ell}\Box K_{m,n}$.

For $k \geq 3$, let $\sigma_t = \lfloor \frac{2\ell(m+n)+t-1}{k} \rfloor$ for $t \in [k]$. We define $S_t = \sum_{s=1}^{t} \sigma_s$ and $T_{t-1} = \sum_{s=2}^{t} \sigma_s$ for $t \in [k]$ and let $T_0 = 0$. Then, $S_t \leq T_t \leq S_t + 1$ and $S_t - T_{t-1} = \sigma_1$ for $t \in [k]$. Since $T_1 = \sigma_2 \leq \sigma_k = \lceil \frac{2\ell(m+n)}{k} \rceil \leq \ell(m+n) < 2\ell(m+n) = S_k$ for $k \geq 3$, there exists $1 < k' \leq k$ such that $T_{k'-1} \leq \ell(m+n) = |\mathcal{V}_1| \leq S_{k'}$. Since $\sigma_1 = \lfloor \frac{2\ell(m+n)}{k} \rfloor \leq (\ell-1)(m+n)$ for $\ell \geq 2$ and $k \geq 3$ except for $\ell = 2$, with $k = 3$, and $\sigma_1 = \lfloor \frac{2\ell(m+n)}{k} \rfloor \leq \ell(m+n) - m$ for $\ell \geq 1$ and $k \geq 3$ except for $\ell = 1$, with $k = 3$, we can partition the vertex set of the product graphs consecutively in their own orderings into k sets of sizes $\sigma_2, \sigma_3, \ldots, \sigma_{k'}, \sigma_1, \sigma_{k'+1}, \sigma_{k'+2}, \ldots, \sigma_k$. Obviously, the k'th set (of size σ_1) is the only possible set containing $(x_{2\ell}, z_n)$ and (x_2, y_1) in $C_{2\ell}\Box K_{m,n}$ (or $(x_{2\ell}, z_n)$ and (x_2, z_1) in $P_{2\ell}\Box K_{m,n}$), and the others are contained in either $\mathcal{V}_1$ or $\mathcal{V}_2$. Hence, the product graphs $C_{2\ell}\Box K_{m,n}$ and $P_{2\ell}\Box K_{m,n}$ are equitably k-colorable, except that $C_4\Box K_{m,n}$ and $P_2\Box K_{m,n}$ may or may not be equitably 3-colorable. □

As $C_4\Box K_{m,n}$ and $P_2\Box K_{m,n}$ may or may not be equitably 3-colorable, we characterize the equitable 3-colorability of $C_4\Box K_{m,n}$ and $P_2\Box K_{m,n}$ below. Notice that $C_4\Box K_{3,3}$ and $P_2\Box K_{3,3}$ are not equitably 3-colorable.

We define $Y_i = \{(x_i, y_j) : j \in [m]\}$ and $Z_i = \{(x_i, z_{j'}) : j' \in [n]\}$ for all i. For any 3-coloring f of $C_4\Box K_{m,n}$ or $P_2\Box K_{m,n}$, a set S of vertices is *c-colored* if $|f(S)| = c$, where $f(S) = \{f(x) : x \in S\}$. Since only three colors can be used, for each i, we have that either Y_i and Z_i are both 1-colored, or one is 1-colored and the other is 2-colored.

Lemma 6. *For positive integers m and n, $C_4\Box K_{m,n}$ is equitably 3-colorable if and only if $m + n + 2 \geq 3\min\{m, n\}$.*

Proof. Without loss of generality, we may assume that $m \leq n$ and so $n \geq 2m - 2$.

($\Leftarrow$) Let $a = m - \lfloor \frac{m+n+2}{3} \rfloor + \lfloor \frac{m+n}{3} \rfloor$ and $b = \lfloor \frac{m+n}{3} \rfloor$. Then, $n \geq b \geq a \geq 0$. For $i \in [4]$, $j \in [m]$ and $j' \in [n]$, let

$$f(x_i, y_j) = \begin{cases} 1, & \text{if } i = 1; \\ 2, & \text{if } i = 3; \\ 3, & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x_i, z_{j'}) = \begin{cases} 1, & \text{if } i = 2, 4 \text{ and } j' \leq b, \text{ or } i = 3 \text{ and } j' > b; \\ 2, & \text{if } i = 2, 4 \text{ and } j' > b, \text{ or } i = 1 \text{ and } j' \leq a; \\ 3, & \text{otherwise;} \end{cases}$$

see Fig. 4. It is straightforward to check that f is a 3-coloring of $C_4\Box K_{m,n}$ with $|f^{-1}(t)| = \lfloor \frac{4(m+n)+t-1}{3} \rfloor$ for $t \in [3]$. Hence, $C_4\Box K_{m,n}$ is equitably 3-colorable.

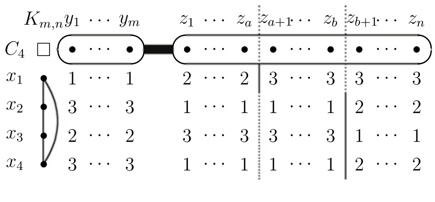

Fig. 4. Equitable 3-coloring of $C_4\Box K_{m,n}$ with $m + n + 2 \geq 3\min\{m, n\}$.

($\Rightarrow$) Suppose to the contrary that $C_4\Box K_{m,n}$ has an equitable 3-coloring f but $n < 2m - 2$. In this case, $2n < \lfloor \frac{4(m+n)}{3} \rfloor \leq \lfloor \frac{4(m+n)+2}{3} \rfloor < 2m + n$. Hence, it is impossible that $|f^{-1}(t)| \geq 2m + n$ or $|f^{-1}(t)| \leq 2n$. Now we let $p = |\{i \in [4]:$

Y_i is 1-colored }| and $q = |\{i \in [4]: Z_i$ is 1-colored }|; then we have $0 \le p, q \le 4$ and $p + q \ge 4$. We consider the following three cases.

Case 1. $p = 4$ (and similarly for $q = 4$). First suppose $\bigcup_{i=1}^{4} Y_i$ is 3-colored, say $f(Y_1) = f(Y_3) = \{1\}, f(Y_2) = \{2\}$ and $f(Y_4) = \{3\}$. Then, $f(x_4, z_{j'}) = 1$ when $f(x_1, z_{j'}) = 2$ and $f(x_2, z_{j'}) = 1$ when $f(x_1, z_{j'}) = 3$, which imply $|f^{-1}(1)| \ge 2m + n$. So now we may assume that $\bigcup_{i=1}^{4} Y_i$ is 2-colored, say $f(Y_1) = f(Y_3) = \{1\}$ and $f(Y_2) = f(Y_4) = \{2\}$. Then, $f^{-1}(3) \subseteq Z_1 \cup Z_2 \cup Z_3 \cup Z_4$ with $\{j' \in [n]: f(x_1, z_{j'}) = 3\} \cap \{j' \in [n]: f(x_2, z_{j'}) = 3\} = \emptyset = \{j' \in [n]: f(x_3, z_{j'}) = 3\} \cap \{j' \in [n]: f(x_4, z_{j'}) = 3\}$ and so $|f^{-1}(3)| \le 2n$.

Case 2. $p = 3$ (and similarly for $q = 3$). We may assume that Y_1, Y_2 and Y_3 are 1-colored and Y_4 is 2-colored, which give $f(Y_1) = f(Y_3)$, say $f(Y_1) = f(Y_3) = \{1\}, f(Y_2) = \{2\}$ and $f(Y_4) = \{2, 3\}$. Then, $f(Z_4) = \{1\}$ and so $|f^{-1}(1)| \ge 2m + n$.

Case 3. $p = q = 2$. We may assume that either Y_1 and Y_3 are 1-colored, or Y_1 and Y_2 are 1-colored. If Y_1 and Y_3 are 1-colored, then $f(Y_1) = f(Y_3)$, say $f(Y_1) = f(Y_3) = \{1\}$ and $f(Y_2) = \{2, 3\}$. Hence, $f(Z_2) = \{1\}$ and so $|f^{-1}(1)| \ge 2m + n$. If Y_1 and Y_2 are 1-colored, say $f(Y_1) = \{1\}$ and $f(Y_2) = \{2\}$, then $f^{-1}(3) \subseteq Z_1 \cup Z_2 \cup Y_3 \cup Y_4$ with $\{j' \in [n]: f(x_1, z_{j'}) = 3\} \cap \{j' \in [n]: f(x_2, z_{j'}) = 3\} = \emptyset = \{j \in [m]: f(x_3, y_j) = 3\} \cap \{j \in [m]: f(x_4, y_j) = 3\}$ and so $|f^{-1}(3)| \le m + n \le 2n$.

In any case, this leads to a contradiction. Therefore, $n \ge 2m - 2$ or equivalently $m + n + 2 \ge 3\min\{m, n\}$. □

Lemma 7. *For positive integers m and n, $P_2 \Box K_{m,n}$ is equitably 3-colorable if and only if $m + n + 2 \ge 3\min\{m, n\}$.*

Proof. Without loss of generality, we may assume that $m \le n$ and so $n \ge 2m - 2$.

($\Leftarrow$) Let $a = m + n - \lfloor \frac{2(m+n)+1}{3} \rfloor$ and $b = \lfloor \frac{2(m+n)+2}{3} \rfloor - m$. Then, $n \ge b \ge a \ge 0$. For $i \in [2], j \in [m]$ and $j' \in [n]$, let

$$f(x_i, y_j) = \begin{cases} 3, & \text{if } i = 1; \\ 2, & \text{if } i = 2, \end{cases} \quad \text{and} \quad f(x_i, z_{j'}) = \begin{cases} 3, & \text{if } i = 2 \text{ and } j' \le b; \\ 2, & \text{if } i = 1 \text{ and } j' > a; \\ 1, & \text{otherwise;} \end{cases}$$

see Fig. 5. It is straightforward to check that f is a 3-coloring of $P_2 \Box K_{m,n}$ with $|f^{-1}(t)| = \lfloor \frac{2(m+n)+t-1}{3} \rfloor$ for $t \in [3]$; hence, $P_2 \Box K_{m,n}$ is equitably 3-colorable.

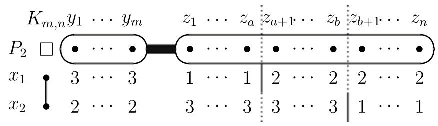

Fig. 5. Equitable 3-coloring of $P_2 \Box K_{m,n}$ with $m + n + 2 \ge 3\min\{m, n\}$.

($\Rightarrow$) Suppose to the contrary that $P_2 \Box K_{m,n}$ has an equitable 3-coloring f but $n < 2m - 2$. In this case, $n < \lfloor \frac{2(m+n)}{3} \rfloor \le \lfloor \frac{2(m+n)+2}{3} \rfloor < m + n$. Hence, it is impossible that $|f^{-1}(i)| \ge m + n$ or $|f^{-1}(i)| \le n$. We consider the following two cases.

Case 1. Y_1 and Y_2 (similarly Z_1 and Z_2) are 1-colored. We may assume that $f(Y_1) = \{1\}$ and $f(Y_2) = \{2\}$. Then, $f^{-1}(3) \subseteq Z_1 \cup Z_2$ with $\{j' \in [n]: f(x_1, z_{j'}) = 3\} \cap \{j' \in [n]: f(x_2, z_{j'}) = 3\} = \emptyset$ and so $|f^{-1}(3)| \le n$.

Case 2. Y_1, Z_2, Y_2 and Z_1 (similarly Y_2, Z_1, Y_1 and Z_2) are 1-, 1-, 2- and 2-colored, respectively. We may assume that $f(Y_1) = \{1\}$ and $f(Y_2) = \{2, 3\}$. Then, $f(Z_2) = \{1\}$ and so $|f^{-1}(1)| \ge m + n$.

In any case, this leads to a contradiction. Therefore, $n \ge 2m - 2$ or equivalently $m + n + 2 \ge 3\min\{m, n\}$. □

According to Lemmas 5–7, we have the following theorem.

Theorem 8. *If ℓ, m and n are positive integers, then $\chi_=(C_{2\ell+2} \Box K_{m,n}) = \chi_=^*(C_{2\ell+2} \Box K_{m,n}) = \chi_=(P_{2\ell} \Box K_{m,n}) = \chi_=^*(P_{2\ell} \Box K_{m,n}) = 2$ except for $\chi_=^*(C_4 \Box K_{m,n}) = \chi_=^*(P_2 \Box K_{m,n}) = 4$, when $m + n + 2 < 3\min\{m, n\}$.*

3.3. The product of two complete bipartite graphs

Finally, we study the Cartesian product of two complete bipartite graphs.

In the following theorems and corollaries, we assume that the bipartitions of $K_{m,n}$ consist of $\{x_i: i \in [m]\}$ and $\{y_j: j \in [n]\}$, and the bipartitions of $K_{m',n'}$ consist of $\{x'_{i'}: i' \in [m']\}$ and $\{y'_{j'}: j' \in [n']\}$. Let $X_1 = \{(x_i, x'_{i'}): i \in [m], i' \in [m']\}, X_2 = \{(y_j, x'_{i'}): j \in [n], i' \in [m']\}, X_3 = \{(x_i, y'_{j'}): i \in [m], j' \in [n']\}$ and $X_4 = \{(y_j, y'_{j'}): j \in [n], j' \in [n']\}$. Notice that $X_1 \cup X_4$ and $X_2 \cup X_3$ are independent sets.

Since $K_{1,1} \cong P_2$ and $K_{1,2} \cong P_3$, we only discuss the case where $m + n \ge 4$ and $m' + n' \ge 4$.

Theorem 9. *If m, n, m' and n' are positive integers such that $m \le n$, $m' \le n'$, $m + n \ge 4$ and $m' + n' \ge 4$, then $K_{m,n} \Box K_{m',n'}$ is equitably k-colorable for $k \ge \lceil \frac{(m+n)(m'+n')}{\max\{m(n'-1), m'(n-1)\}+1} \rceil$.*

Proof. Without loss of generality, assume that $m'(n-1) \le m(n'-1)$. Let $\sigma_t = \lfloor \frac{(m+n)(m'+n')+t-1}{k} \rfloor$ for $t \in [k]$. By the assumption, $k \ge \lceil \frac{(m+n)(m'+n')}{m(n'-1)+1} \rceil \ge \frac{(m+n)(m'+n')}{m(n'-1)+1}$ and so $\sigma_t \le \sigma_k = \lceil \frac{(m+n)(m'+n')}{k} \rceil \le m(n'-1) + 1 \le mn'$.

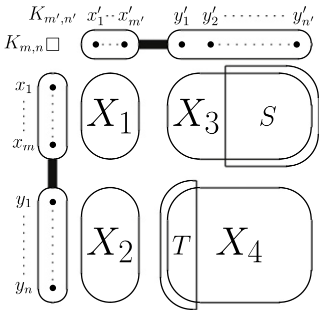

Fig. 6. Choice of an independent σ_k-set $S \cup T \subseteq X_3 \cup X_4$ in $K_{m,n}\square K_{m',n'}$.

Since $\sigma_1 \le m'n + mn' \le (m+n)(m'+n') - mn' \le \sum_{t<k} \sigma_t$, there exists $1 < k' < k$ such that $\sum_{t<k'} \sigma_t \le m'n + mn' = |X_2 \cup X_3| < \sum_{t\le k'} \sigma_t$. Let $\sigma = m'n + mn' - \sum_{t<k'} \sigma_t$. Then $0 \le \sigma < \sigma_{k'} \le m(n'-1) + 1$. Choose a set S consisting of σ vertices $(x_i, y'_{j'}) \in X_3$ with j' as large as possible, and choose a set T consisting of $(\sigma_{k'} - \sigma)$ vertices $(y_j, y'_{j'}) \in X_4$ with j' as small as possible, see Fig. 6:

Since $\lceil \frac{\sigma}{m} \rceil \le \lceil \frac{m(n'-1)}{m} \rceil < n'$ and $\lceil \frac{\sigma_{k'}-\sigma}{n} \rceil \le \lceil \frac{\sigma_{k'}-\sigma}{m} \rceil \le \lceil \frac{m(n'-1)+1-\sigma}{m} \rceil \le n' - \lceil \frac{\sigma}{m} \rceil$, S and T can be found such that $S \cup T$ is an independent set. We can partition $(X_2 \cup X_3) \setminus S$ into $(k'-1)$ sets of sizes $\sigma_1, \sigma_2, \ldots, \sigma_{k'-1}$, and partition $(X_1 \cup X_4) \setminus T$ into $(k-k')$ sets of sizes $\sigma_{k'+1}, \sigma_{k'+2}, \ldots, \sigma_k$. Hence, these $(k-1)$ sets are independent and together with $S \cup T$ yield an equitable k-coloring of $K_{m,n}\square K_{m',n'}$. □

Corollary 10. *If we have integers $m \ge 3$ and $n \ge 3$, then $K_{1,m}\square K_{1,n}$ is equitably k-colorable for $k \ge \min\{m,n\} + 2$.*

Proof. Applying Theorem 9, we have that $K_{1,m}\square K_{1,n}$ is equitably k-colorable for $k \ge \lceil \frac{(m+1)(n+1)}{\max\{m,n\}} \rceil = \min\{m,n\} + 2$ except for the case of $m = n$, with $k = m + 2$. When $m = n$, for $t \in [m+2]$, we have $\sigma_t = \lfloor \frac{(m+1)^2+t-1}{m+2} \rfloor \le m$ except for $\sigma_{m+2} = \lceil \frac{(m+1)^2}{m+2} \rceil = m+1$, and hence $\sigma_1 \le 2m \le (m+1)^2 - (m+1) \le \sum_{t<m+2} \sigma_t$. So, we can use the same process as in Theorem 9 to give an equitable $(m+2)$-coloring of $K_{1,m}\square K_{1,m}$. □

Theorem 11. *If m, n, m' and n' are positive integers, then $K_{m,n}\square K_{m',n'}$ is equitably 4-colorable.*

Proof. Assume that $m \le n$ and $m' \le n'$. Let $\sigma_t = \lfloor \frac{(m+n)(m'+n')+t-1}{4} \rfloor$ for $t \in [4]$.

If we can find two independent sets S and T with $|S| = \sigma_2$ and $|T| = \sigma_3$ such that $X_2 \cup X_3 \subseteq S \cup T$, then since $X_1 \cup X_4$ and $X_2 \cup X_3$ are independent we can partition the other vertices into two sets of sizes σ_1 and σ_4. These four sets then yield an equitable 4-coloring of $K_{m,n}\square K_{m',n'}$.

We consider the following four sets (see Fig. 7 for an example):

$$\mathcal{V}_1 = \left\{(x_i, y'_{j'}) : 1 \le m(j'-1) + i \le \left\lfloor \frac{mn'}{2} \right\rfloor \right\} \cup \left\{(y_j, x'_{i'}) : 1 \le m'(j-1) + i' \le \left\lceil \frac{nm'}{2} \right\rceil \right\},$$

$$\mathcal{V}_2 = \left\{(x_i, y'_{j'}) : \left\lfloor \frac{mn'}{2} \right\rfloor < m(j'-1) + i \le mn' \right\} \cup \left\{(y_j, x'_{i'}) : \left\lceil \frac{nm'}{2} \right\rceil < m'(j-1) + i' \le nm' \right\},$$

$$\mathcal{U}_1 = \left\{(y_j, y'_{j'}) : \left\lceil \frac{n}{2} \right\rceil < j \le n, \left\lceil \frac{n'}{2} \right\rceil < j' \le n' \right\} \quad \text{and}$$

$$\mathcal{U}_2 = \left\{(y_j, y'_{j'}) : 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor, 1 \le j' \le \left\lfloor \frac{n'}{2} \right\rfloor \right\}.$$

Notice that $\mathcal{V}_1, \mathcal{V}_2, \mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1 \cup \mathcal{U}_1$ and $\mathcal{V}_2 \cup \mathcal{U}_2$ are independent. Since $m \le n$ and $m' \le n'$, we have

$$\begin{aligned} |\mathcal{V}_1 \cup \mathcal{U}_1| + |\mathcal{V}_2 \cup \mathcal{U}_2| &= mn' + nm' + 2\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n'}{2} \right\rfloor \ge \left\lceil \frac{(m+n)(m'+n')}{2} \right\rceil \\ &\ge \sigma_2 + \sigma_3 \ge \left\lfloor \frac{(m+n)(m'+n')}{2} \right\rfloor \ge mn' + nm' = |\mathcal{V}_1| + |\mathcal{V}_2| \end{aligned}$$

and

$$0 \le \big||\mathcal{V}_1 \cup \mathcal{U}_1| - |\mathcal{V}_2 \cup \mathcal{U}_2|\big| = \big||\mathcal{V}_1| - |\mathcal{V}_2|\big| \le 1, \quad \text{and} \quad 0 \le \sigma_3 - \sigma_2 \le 1.$$

Then, $|\mathcal{V}_1 \cup \mathcal{U}_1| \ge \sigma_s \ge |\mathcal{V}_1|$ and $|\mathcal{V}_2 \cup \mathcal{U}_2| \ge \sigma_t \ge |\mathcal{V}_2|$ for $\{s,t\} = \{2,3\}$, and hence we can choose $\mathcal{V}_1 \subseteq S \subseteq \mathcal{V}_1 \cup \mathcal{U}_1$ and $\mathcal{V}_2 \subseteq T \subseteq \mathcal{V}_2 \cup \mathcal{U}_2$ with $|S| = \sigma_s$ and $|T| = \sigma_t$ as desired. □

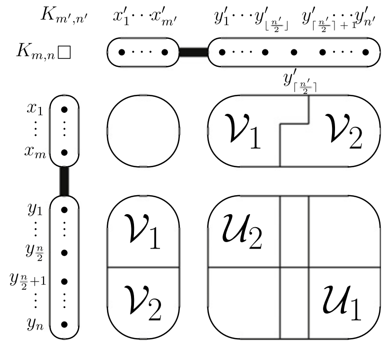

Fig. 7. A vertex partition of $K_{m,n}\Box K_{m',n'}$ with even n and odd n'.

Corollary 12. *If we have integers $m \geq 3$ and $n \geq 3$, then $K_{1,m}\Box K_{1,n}$ is equitably k-colorable for $4 \leq k \leq \min\{m, n\} + 1$.*

Proof. For $4 \leq k \leq \min\{m, n\} + 1$, let $\sigma_t = \lfloor \frac{(m+1)(n+1)+t-1}{k} \rfloor$ for $t \in [k]$. Also, if we can find two independent sets S and T with $|S| = \sigma_1$ and $|T| = \sigma_2$ such that $X_2 \cup X_3 \subseteq S \cup T$, then we can partition the other vertices into $k - 2$ sets of sizes $\sigma_3, \sigma_4, \ldots, \sigma_k$. These k sets yield an equitable k-coloring of $K_{1,m}\Box K_{1,n}$.

We consider the same four sets $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{U}_1$ and $\mathcal{U}_2$ as in Theorem 11. Since $4 \leq k \leq \min\{m, n\} + 1$, we have

$$|\mathcal{V}_1 \cup \mathcal{U}_1| + |\mathcal{V}_2 \cup \mathcal{U}_2| = m + n + 2\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \geq \left\lfloor \frac{(m+1)(n+1)}{2} \right\rfloor$$
$$\geq \left\lfloor \frac{2(m+1)(n+1)}{k} \right\rfloor \geq \sigma_1 + \sigma_2 \geq 2\left\lfloor \frac{(m+1)(n+1)}{k} \right\rfloor \geq m + n = |\mathcal{V}_1| + |\mathcal{V}_2|.$$

Similarly, we have $|\mathcal{V}_1 \cup \mathcal{U}_1| \geq \sigma_s \geq |\mathcal{V}_1|$ and $|\mathcal{V}_2 \cup \mathcal{U}_2| \geq \sigma_t \geq |\mathcal{V}_2|$ for $\{s, t\} = \{1, 2\}$. Hence, we can choose $\mathcal{V}_1 \subseteq S \subseteq \mathcal{V}_1 \cup \mathcal{U}_1$ and $\mathcal{V}_2 \subseteq T \subseteq \mathcal{V}_2 \cup \mathcal{U}_2$ with $|S| = \sigma_s$ and $|T| = \sigma_t$ as desired. $\Box$

For the case of the Cartesian product of two stars, we shall characterize the equitable 3-colorability of $K_{1,m}\Box K_{1,n}$. For convenience, let the bipartition of $K_{1,m}$ consist of $\{x_0\}$ and $\{x_i : i \in [m]\}$, and the bipartition of $K_{1,n}$ consist of $\{y_0\}$ and $\{y_j : j \in [n]\}$.

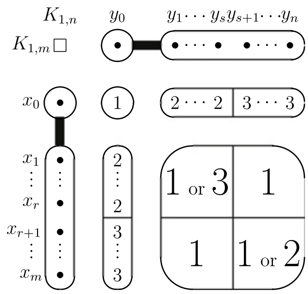

Fig. 8. Equitable 3-coloring of $K_{1,m}\Box K_{1,n}$ with $(m-2)(n-2) > 5$.

Lemma 13. *For integers $m \geq 3$ and $n \geq 3$, $K_{1,m}\Box K_{1,n}$ is equitably 3-colorable if and only if $(m-2)(n-2) \leq 5$.*

Proof. ($\Leftarrow$) First, $K_{1,m}\Box K_{1,n}$ is a bipartite graph whose unique partition is given as $X_2 \cup X_3$ and $X_1 \cup X_4$ with $|X_2 \cup X_3| = m + n$ and $|X_1 \cup X_4| = mn + 1$. Since $m \geq 3$, $n \geq 3$ and $(m-2)(n-2) \leq 5$, we have $m + n - 1 \leq \frac{mn+1}{2} \leq m + n + 1$. Now we can partition $X_1 \cup X_4$ into two sets of sizes $\lfloor \frac{mn+1}{2} \rfloor$ and $\lceil \frac{mn+1}{2} \rceil$. Hence, the two sets are independent and together with $X_2 \cup X_3$ yield an equitable 3-coloring of $K_{1,m}\Box K_{1,n}$.

($\Rightarrow$) Suppose to the contrary that $K_{1,m}\Box K_{1,n}$ has an equitable 3-coloring f but $(m-2)(n-2) > 5$. We may assume that $f(x_0, y_0) = 1$, $f(x_i, y_0) = f(x_0, y_j) = 2$ for $1 \leq i \leq r$ and $1 \leq j \leq s$, and $f(x_i, y_0) = f(x_0, y_j) = 3$ for $r < i \leq m$ and $s < j \leq n$, for some $0 \leq r \leq m$ and $0 \leq s \leq n$, by renaming vertices. The colors of the other vertices are forced as shown in Fig. 8. Let $r' = m - r$ and $s' = n - s$. Here we consider three cases.

Case 1. $r \leq 1$ or $s \leq 1$ (resp. $r' \leq 1$ or $s' \leq 1$). In this case, $|f^{-1}(3)|$(resp. $|f^{-1}(2)|$) $\leq m + n < \lfloor \frac{(m+1)(n+1)}{3} \rfloor$.

By the preceding case, we have $1 < r, r' < m - 1$ and $1 < s, s' < n - 1$, and hence $m \geq 4$ and $n \geq 4$. By symmetry, we may assume $r \leq r'$.

Case 2. $1 < r \le r' < m-1$ and $1 < s' < s < n-1$. In this case, $|f^{-1}(1)| - |f^{-1}(2)| \ge (1+rs'+r's) - (r+s+r's') = (r'-1)(s-s') + (r-1)(s'-1) \ge (r'-1) + (r-1)(s'-1) > 1$.

Case 3. $1 < r \le r' < m-1$ and $1 < s \le s' < n-1$. In this case, $|f^{-1}(1)| - |f^{-1}(3)| \ge (1+rs'+r's) - (r'+s'+rs) = (r'-r)(s-1) + (r-1)(s'-1) = \begin{cases} (r-1)(s'-1) \ge \left(\frac{m}{2}-1\right)\left(\frac{n}{2}-1\right), & \text{if } r'=r=\frac{m}{2}; \\ (s-1)+(r-1)(s'-1) \ge 2, & \text{if } r' > r \end{cases} > 1$.

In any case, this leads to a contradiction. Therefore, $K_{1,m} \Box K_{1,n}$ is not equitably 3-colorable. $\Box$

According to Corollaries 10 and 12 and Lemma 13, we have the following theorem.

Theorem 14. *If we have integers $m \ge 3$ and $n \ge 3$, then $\chi_=(K_{1,m} \Box K_{1,n}) = \chi^*_=(K_{1,m} \Box K_{1,n}) = 4$ except for $\chi_=(K_{1,m} \Box K_{1,n}) = \chi^*_=(K_{1,m} \Box K_{1,n}) = 3$, when $(m-2)(n-2) \le 5$.*

Proof. Since $K_{1,m} \Box K_{1,n}$ is a bipartite graph whose partition sets are determined in only one way and differ in size by $|(mn+1)-(m+n)| = (m-1)(n-1) \ge 4$, it is not equitably 2-colorable and so $\chi_=(K_{1,m} \Box K_{1,n}) \ge 3$. The theorem then follows. $\Box$

We remark that Theorem 14 shows that the gap in the inequality $\chi^*_=(G \Box H) \le \max\{\chi^*_=(G), \chi^*_=(H)\}$ of Corollary 2 can be arbitrarily large, as shown by the examples $\chi^*_=(K_{1,m} \Box K_{1,n}) = 4 < 1 + \lceil \frac{n}{2} \rceil = \chi^*_=(K_{1,n}) = \max\{\chi^*_=(K_{1,m}), \chi^*_=(K_{1,n})\}$ for $n \ge m \ge 7$.

4. Conclusion

In this paper, we obtain the following results for positive integers ℓ, $m \le n$, $m' \le n'$ and a bipartite graph H.

1. $\chi_=(C_{2\ell+1} \Box H) = \chi^*_=(C_{2\ell+1} \Box H) = \chi_=(P_{2\ell+1} \Box H) = \chi^*_=(P_{2\ell+1} \Box H) = 3$ except for $\chi_=(P_{2\ell+1} \Box H) = \chi^*_=(P_{2\ell+1} \Box H) = 2$, when $\chi_=(H) \le 2$.
2. $\chi_=(C_{2\ell+2} \Box K_{m,n}) = \chi^*_=(C_{2\ell+2} \Box K_{m,n}) = \chi_=(P_{2\ell} \Box K_{m,n}) = \chi^*_=(P_{2\ell} \Box K_{m,n}) = 2$ except for $\chi^*_=(C_4 \Box K_{m,n}) = \chi^*_=(P_2 \Box K_{m,n}) = 4$, when $m+n+2 < 3\min\{m,n\}$.
3. $K_{m,n} \Box K_{m',n'}$ is equitably k-colorable for $k \ge \lceil \frac{(m+n)(m'+n')}{\max\{m(n'-1), m'(n-1)\}+1} \rceil$.
4. $K_{m,n} \Box K_{m',n'}$ is equitably 4-colorable.
5. $\chi_=(K_{1,m+2} \Box K_{1,n+2}) = \chi^*_=(K_{1,m+2} \Box K_{1,n+2}) = 4$ except for $\chi_=(K_{1,m+2} \Box K_{1,n+2}) = \chi^*_=(K_{1,m+2} \Box K_{1,n+2}) = 3$, when $mn \le 5$.

For any bipartite graph G, the product graphs $C_4 \Box G$ and $P_2 \Box G$ are equitably k-colorable for $k = 2$ or $k \ge 4$, and are equitably 3-colorable if G is a subgraph of $K_{m,n}$ for positive integers m and n with $m+n+2 \ge 3\min\{m,n\}$. Also, $C_4 \Box K_{m,n}$ and $P_2 \Box K_{m,n}$ are not equitably 3-colorable for $m \le n < 2m-2$. We pose the following question.

Problem 1. For a proper subgraph G of $K_{m,n}$ with $m \le n < 2m-2$, what are the conditions such that $C_4 \Box G$ or $P_2 \Box G$ is equitably 3-colorable?

While $\chi_=(G \Box H) \le 4$ for bipartite graphs G and H, we believe that this upper bound is also true for the equitable chromatic threshold.

Conjecture 2. *$\chi^*_=(G \Box H) \le 4$ for bipartite graphs G and H.*

Conjecture 3. *$\chi_=(G \Box H) \le \chi(G)\chi(H)$ for connected graphs G and H.*

Notice that $\chi_=(K_{1,3} \Box \overline{K_3}) = \chi^*_=(K_{1,3} \Box \overline{K_3}) = 3 > 2 = \chi(K_{1,3})\chi(\overline{K_3})$, where $\overline{G}$ is the complement graph of G. Hence, the connectivity of graphs in Conjecture 3 is necessary.

Acknowledgments

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